Lecture 26: Fourier Transform Teaser

- In a "normal" vector space like $\mathbb{R}^n$, we come up with tricks like diagonalization to simplify calculations. How do you diagonalize a partial differential operator?

  → We focus on functions, which we normally consider pointwise like $\{f(y)\}_{y\in\mathbb{R}^n}$. We can think of this as acting under the "basis" $\{\delta(x-y)\}_{y\in\mathbb{R}^n}$ in the sense that $f(x) = \int f(y)\,\delta(x-y)\,dy$ (for $f\in C^0(\mathbb{R}^n)$).
  or "f's coefficient at x is $\langle f, \delta(x-\cdot)\rangle$"
  So that $\{\delta(x-y)\}$ uniquely determines f via its coefficients ($\delta(x-y)$ is "linearly independent")

  → We want a basis in which differentiation looks diagonal, so we might pick
  $$\{e^{i\xi\cdot x}\}_{\xi\in\mathbb{R}^n}$$
  because
  $$\partial_j e^{i\xi\cdot x} = i\xi_j\, e^{i\xi\cdot x}$$
  but, is this still a "basis"?
  Remarkably, yes! Given a function f on $\mathbb{R}^d$ ($f\in L^2$), we can write f uniquely in the form
  $$f(x) = \int a(\xi)\, e^{i\xi\cdot x}\,dx$$

- The Fourier transform $\mathcal{F}$ is the "change-of-basis"
  $f \mapsto a(\xi)$

- Let us try to find the form of $\mathcal{F}$. We expect from linear algebra $\hat{f} = Af$ for a matrix A, vector b,

So here we try

$$\mathcal{F}[f](\xi) \propto \int f(y) m(y,\xi) dy$$

to match the above processes. Then,

$$\delta_0(x-y) \text{ "=" } \int m(y,\xi) e^{i\xi\cdot x} d\xi$$

- By translation symmetry, we expect

$$\int m(y,\xi) e^{i\xi\cdot x} dx = \delta_0(x-y) = \int m(0,\xi) e^{i\xi\cdot(x-y)} d\xi$$

So that if we believe $\{e^{i\xi\cdot x}\}$ forms a basis,

$$m(0,\xi) e^{-i\xi\cdot y} = m(y,\xi)$$

- Next, $\delta(x)$ has the property $x^j \delta(x) = 0$, while

$$0 = x^j \delta(x) = \int m(0,\xi) x^j e^{i\xi\cdot x} d\xi =$$

$$\int m(0,\xi) \tfrac{1}{i} \partial_{\xi_j} e^{i\xi\cdot x} dx$$

$$= i \int \partial_\xi (m(0,\xi)) e^{i\xi\cdot x} dx$$

so $\partial_\xi m(0,\xi)$ is a nonzero constant, or

$$\delta(x) = c \int e^{i\xi\cdot x} dx$$

and $m(y,\xi) \propto e^{-i\xi\cdot y}$

Thus, we ~~claim~~ see

$$\mathcal{F}[f](\xi) \propto \int f(y) e^{-i\xi\cdot y} dy$$

and claim the constant of proportionality is $c = \frac{1}{(2\pi)^n}$.

• Approaching via approximation

First, $\delta(x) = \delta(x_1) \dots \delta(x_n)$, so

$$\int e^{ix\cdot\xi} d\xi = \prod_{i=1}^{n} \int e^{i\xi_i x_i} d\xi_i$$

and $c = (c_1)^n$. We consider the 1D case

To make sense of $\int_{\mathbb{R}} e^{i\xi\cdot x} dx$, we "temper" by multiplying by $e^{-\varepsilon|\xi|}$, integrate, and take $\varepsilon \to 0$:

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} e^{i\xi x - \varepsilon|\xi|} d\xi$$

$$= \lim_{\varepsilon\to 0^+} \int_{-\infty}^{0} e^{i\xi(x-i\varepsilon)} d\xi + \int_0^\infty e^{i\xi(x+i\varepsilon)} d\xi$$

$$= \lim_{\varepsilon\to 0^+} \left[\frac{1}{i(x-i\varepsilon)} - \frac{1}{i(x+i\varepsilon)} \right]$$

$$= \lim_{\varepsilon\to 0} \frac{2\varepsilon}{x^2+\varepsilon^2} = 2\left(\int \frac{dt}{1+t^2}\right)\delta_0 = 2\pi\,\delta_0$$

by our distributions yesterday

and so $c_1 = \frac{1}{2\pi}$, $c = \frac{1}{(2\pi)^n}$.

• In general, we define

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(y) e^{-i\xi\cdot y} dy$$

for $f \in C_c^\infty(\mathbb{R}^n; \mathbb{C})$

in fact,

$$f(x) = c\mathcal{F}^*(\hat{f})(x) = c\int_{\mathbb{R}^n} \hat{f}(\xi) e^{i\xi\cdot x} d\xi$$

for $c = \frac{1}{(2\pi)^n}$

• The fourier transform has many useful properties, but foremost to PDE's is

$$\mathcal{F}[\partial_j f](\xi) = i\xi_j \mathcal{F}[f](\xi)$$

$$\mathcal{F}[x_j f](\xi) = i\partial_{\xi_j} \mathcal{F}[f](\xi)$$

• This allows us to "mess with" PDE's. Consider

$$\begin{cases} (\partial_t - \Delta)u = 0 \\ u(0,x) = g \end{cases}$$

if we take the Fourier transform only in <u>space</u>, this becomes

$$\begin{cases} \partial_t \hat{u}(t,\xi) + |\xi|^2 \hat{u}(t,\xi) = 0 \\ \hat{u}(0,\xi) = \hat{g}(\xi) \end{cases}$$

with solution $\hat{u}(t,\xi) = e^{-t|\xi|^2}\hat{g}(\xi)$

So solving the heat equation simply means transforming back. Another property of the transform is

$$\mathcal{F}[f*g] = \mathcal{F}[f]\mathcal{F}[g], \text{ so}$$

$$\mathcal{F}^{-1}[e^{-t|\xi|^2}\hat{g}(\xi)] = \mathcal{F}^{-1}[e^{-t|\xi|^2}] * \mathcal{F}^{-1}[\hat{g}]$$

$$= (4\pi t)^{-n/2} e^{-|x|^2/4t} * g(x) \, !$$